On the Entropy of Random Fibonacci Words

Johan Nilsson

Centre for Mathematical Sciences, Lund University, LTH

Box 118, SE-221 00 LUND, SWEDEN

`johann@maths.lth.se`

Abstract

The random Fibonacci chain is a generalisation of the classical Fibonacci substitution and is defined as the rule mapping $0 \mapsto 1$ and $1 \mapsto 01$ with probability p and $1 \mapsto 10$ with probability $1 - p$ for $0 < p < 1$ and where the random rule is applied each time it acts on a 1. We show that the topological entropy of this object is given by the growth rate of the set of inflated random Fibonacci words.

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1 Introduction

In [2] Godrèche and Luck define the random Fibonacci chain by the generalised substitution

$$\theta : \begin{cases} 0 & \mapsto & 1 \\ 1 & \mapsto & \begin{cases} 01 & \text{with probability } p \\ 10 & \text{with probability } 1 - p \end{cases} \end{cases} \tag{1}$$

for $0 < p < 1$ and where the random rule is applied each time θ acts on a 1. They introduce the random Fibonacci chain when studying quasi-crystalline structures and tilings in the plane. In their paper it is claimed without proof that the topological entropy of the random Fibonacci chain is given by the growth rate of the set of inflated random Fibonacci words. We give here a proof of this fact.

Before we can state our main theorem we need to introduce some notation. A word w over an alphabet Σ is a finite sequence $w_1 w_2 \ldots w_n$ of symbols from Σ. We let here $\Sigma = \{0, 1\}$. We denote a sub-word or a factor of w by $w[a, b] = w_a w_{a+1} w_{a+2} \ldots w_{b-1} w_b$ and similarly we let $W[a, b] = \{w[a, b] : w \in W\}$. By $|\cdot|$ we mean the length of a word and the cardinality of a set. Note that $|w[a, b]| = b - a + 1$.

For two words $u = u_1 u_2 u_3 \ldots u_n$ and $v = v_1 v_2 v_3 \ldots v_m$ we denote by uv the concatenation of the two words, that is, $uv = u_1 u_2 u_3 \ldots u_n v_1 v_2 \ldots v_m$. Similarly we let for two sets of words U and V their product be the set $UV = \{uv : u \in U, v \in V\}$ containing all possible concatenations.

Letting θ act on the word 0 repeatedly yields us an infinite sequence of words $r_n = \theta^{n-1}(0)$. We know that $r_1 = 0$ and $r_2 = 1$. But r_3 is either 01 or

10, with probability p or $1-p$ respectively. The sequence $\{r_n\}_{n=1}^{\infty}$ converges in distribution to an infinite random word r. We say that r_n is an inflated word (under θ) in generation n and we introduce here sets that correspond to all inflated words in generation n;

Definition 1. *Let $A_0 := \emptyset$, $A_1 := \{0\}$ and $A_2 := \{1\}$ and for $n \geq 3$ we define recursively*

$$A_n := A_{n-1}A_{n-2} \cup A_{n-2}A_{n-1} \qquad (2)$$

and we let $A := \lim_{n \to \infty} A_n$.

We shall later on, as a direct consequence of Proposition 7, see that the set A is a well defined set. It is clear from the definition of A_n that all elements in A_n have the same length, that is, for all $x, y \in A_n$ we have $|x| = |y|$. We shall frequently us the length of the elements in A_n, and therefore we introduce the notation

$$f_n := |x| \quad \text{for} \quad x \in A_n, \quad n \geq 1$$

and for completeness we set $f_0 := 0$. From the recursion (2) and the definition of f_n we have immediately the following proposition

Proposition 2. *The numbers f_n are the Fibonacci-numbers, that is, $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.*

For a word w we say that x is a factor of w if there are two words u, v such that $w = uxv$. The factor set $F(S, n)$, of a set of words S, is the set of all factors of length n of the words in S. We introduce the abbreviated notation F_n for the special factor sets of the random Fibonacci words,

$$F_n := F(A, f_n).$$

It seems to be a hard problem to give an explicit expression for the size of $|F_n|$. We have to leave that question open, but we shall give a rough upper bound of $|F_n|$.

The topological entropy of the random Fibonacci chain is defined as the limit $\lim_{n \to \infty} \frac{1}{n} \log |F(A, n)|$. The existence of this limit is direct by Fekete's lemma [1] since we have sub-additivity, $\log |F(A, n + m)| \leq \log |F(A, n)| + \log |F(A, m)|$. We can now state the main result in this paper

Theorem 3. *The logarithm of the growth rate of the size of the set of inflated random Fibonacci words equals the topological entropy of the random Fibonacci chain, that is*

$$\lim_{n \to \infty} \frac{\log |A_n|}{f_n} = \lim_{n \to \infty} \frac{\log |F_n|}{f_n}. \qquad (3)$$

The outline of the paper is that we start by studying the set A_n and restate some of the results obtained by Godrèche and Luck. Thereafter we continue by looking at sets of prefixes of the set A_n. Next we give a finite method for finding the factor set F_n and finally we present an estimate of $|F_n|$ in terms of $|A_n|$, leading to the proof of Theorem 3.

2   Inflated Words

In this section we present the sets of inflated random Fibonacci words and give an insight to their structure. For some small values of n we have directly from the definition

$$A_1 = \{0\}, \quad A_2 = \{1\}, \quad A_3 = \{01, 10\}, \quad A_4 = \{011, 101, 110\},$$

$$A_5 = \left\{ \begin{array}{l} 01011, 01101, 01110, 10011, \\ 10101, 10110, 11001, 11010 \end{array} \right\}.$$

For a word x we denote the reversed word of x by x^r, that is, if $x = x_1 x_2 \ldots x_n$ then $x^r = x_n x_{n-1} \ldots x_1$. Similarly we write $A_n^r = \{x^r : x \in A_n\}$.

Proposition 4. *For $n \geq 1$ we have $A_n = A_n^r$.*

Proof. We give a proof by induction on n. The basis cases $n = 1$ and $n = 2$ are clear as $A_1 = \{0\}$ and $A_2 = \{1\}$. Now assume for induction that the statement of the proposition holds for $2 \leq n \leq p$. Using (2) and the induction assumption give

$$\begin{aligned} A_{p+1}^r &= (A_p A_{p-1} \cup A_{p-1} A_p)^r \\ &= A_{p-1}^r A_p^r \cup A_p^r A_{p-1}^r \\ &= A_{p-1} A_p \cup A_p A_{p-1} \\ &= A_{p+1}, \end{aligned}$$

which completes the induction. $\qquad\square$

Now let us turn to the question of the size of A_n. In the recursive definition (2) of A_n we have that the overlap is

$$A_n A_{n-1} \cap A_{n-1} A_n = A_{n-2} A_{n-3} A_{n-2}.$$

Knowing this we obtain the recursion for $n \geq 3$

$$|A_n| = 2|A_{n-1}||A_{n-2}| - |A_{n-2}|^2 |A_{n-3}|, \tag{4}$$

with $|A_0| = 0$, $|A_1| = 1$ and $|A_2| = 1$. The recursion (4) was given by Godrèche and Luck in [2]. They also gave the following proposition, which we for completeness restate here with its short proof.

Proposition 5 ([2])**.** *With $|A_1| = 1$ and $|A_2| = 1$ we have for $n \geq 3$*

$$|A_n| = \frac{n-1}{n-2}|A_{n-1}||A_{n-2}|. \tag{5}$$

Proof. Let $a_n = |A_n|/(|A_{n-1}||A_{n-2}|)$. Then we can rewrite the recursion (4) into

$$a_n = 2 - \frac{1}{a_{n-1}}$$

with the initial condition $a_3 = 2$. We claim that $a_n = \frac{n-1}{n-2}$. The proof of the claim is a straight forward induction. The basis is clear, and if we assume the claim being true for $n = p$ we get for $n = p + 1$

$$a_{p+1} = 2 - \frac{1}{a_p} = 2 - \frac{1}{\frac{p-1}{p-2}} = \frac{p}{p-1},$$

which completes the induction and the proof of the proposition. $\qquad\square$

Without explicitly stating it, the following proposition was also given by Godrèche and Luck in [2]. We present it here, filling in the details.

Proposition 6 ([2]). *For $n \geq 3$ we have*

$$|A_n| = (n-1) \prod_{i=2}^{n-1} (n-i)^{f_{i-2}}. \tag{6}$$

Proof. We give a proof by induction on n. For the basis steps, $n = 3$ and $n = 4$ we have $|A_3| = 2 \cdot 1^0 = 2$ and $|A_4| = 3 \cdot 2^0 \cdot 1^1 = 3$. Now assume for induction that (6) holds for $4 \leq n \leq p$. Then from (5) and the induction assumption we have

$$|A_{p+1}| =$$
$$= \frac{p}{p-1} |A_p||A_{p-1}|$$
$$= \frac{p}{p-1} \left((p-1) \prod_{i=2}^{p-1} (p-i)^{f_{i-2}} \right) \left((p-2) \prod_{i=2}^{(p-1)-1} ((p-1)-i)^{f_{i-2}} \right)$$
$$= p(p-2) \left(\prod_{i=2}^{p-1} (p-i)^{f_{i-2}} \right) \left(\prod_{i=2}^{(p-1)-1} (p-(i+1))^{f_{(i+1)-3}} \right)$$
$$= p(p-2) \left(\prod_{i=2}^{p-1} (p-i)^{f_{i-2}} \right) \left(\prod_{i=3}^{p-1} (p-i)^{f_{i-3}} \right)$$
$$= p(p-2)^{f_1} (p-2)^{f_0} \left(\prod_{i=3}^{p-1} (p-i)^{f_{i-2}+f_{i-3}} \right)$$
$$= p(p-1)^{f_0} (p-2)^{f_1} \left(\prod_{i=3}^{p-1} (p-i)^{f_{i-1}} \right)$$

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$$= p\,(p-1)^{f_0}(p-2)^{f_1}\left(\prod_{i=4}^{p}(p-(i-1))^{f_{(i-1)-1}}\right)$$

$$= p\left(\prod_{i=2}^{(p+1)-1}((p+1)-i)^{f_{i-2}}\right),$$

which completes the induction. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The sequence $\{|A_n|\}_{n=1}^{\infty}$ (see Table 1 on page 11) is the sequence A072042 in the On-line Encyclopedia of Integer Sequences [3].

From Proposition 6 we have

$$\frac{\log|A_n|}{f_n} = \frac{1}{f_n}\log\left((n-1)\prod_{i=2}^{n-1}(n-i)^{f_{i-2}}\right)$$

$$= \frac{1}{f_n}\log(n-1) + \sum_{i=2}^{n-1}\frac{f_{i-2}}{f_n}\log(n-i). \qquad (7)$$

For large n the Fibonacci number f_n can be approximated by $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$. Hence the sum in (7) converges when letting n tend to infinity and therefore clearly implies the existence of the limit. A numerical calculation in Maple gives

$$\lim_{n\to\infty}\frac{\log|A_n|}{f_n} \approx 0.444399 \approx \log 1.559553.$$

3 Prefixes

In this section we look at and present some properties of sets of prefixes of the inflated random Fibonacci words.

Proposition 7. *For $n \geq 3$ and $k \geq 0$ we have*

$$A_n[1, f_n - 1] = A_{n+k}[1, f_n - 1] \qquad (8)$$

and symmetrically

$$A_n[2, f_n] = A_{n+k}[f_{n+k} - f_n + 2, f_{n+k}]. \qquad (9)$$

Proof. We put our initial attention to equality (8). Let us first consider the case with $k = 1$. We will give a proof by induction on n. For the basis we have

$$A_3[1, 1] = \{0, 1\} = A_4[1, 1] \quad \text{and} \quad A_4[1, 2] = \{01, 10, 11\} = A_5[1, 2].$$

Now assume that (8) holds for $4 \leq n \leq m$. In the induction step $n = m+1$ we get

$$A_{m+1}[1, f_{m+1} - 1] \subset (A_{m+1}A_m \cup A_m A_{m+1})[1, f_{m+1} - 1]$$
$$= A_{m+2}[1, f_{m+1} - 1].$$

For the reversed inclusion let $x \in A_{m+2}[1, f_{m+1} - 1]$. By the recursion identity (2) we have to look at two cases. First, if $x \in (A_{m+1}A_m)[1, f_{m+1}-1]$ then we clearly have $x \in A_{m+1}[1, f_{m+1} - 1]$ and we are done. Secondly, we have to deal with the case $x \in (A_m A_{m+1})[1, f_{m+1} - 1]$. For this case we use the induction assumption twice to obtain

$$A_{m-1}[1, f_{m-1} - 1] = A_m[1, f_{m-1} - 1]$$
$$= \big(A_m[1, f_m - 1]\big)[1, f_{m-1} - 1]$$
$$= \big(A_{m+1}[1, f_m - 1]\big)[1, f_{m-1} - 1]$$
$$= A_{m+1}[1, f_{m-1} - 1].$$

By the help of the above equality it follows that

$$\big(A_m A_{m+1}\big)[1, f_{m+1} - 1] = A_m\big(A_{m+1}[1, f_{m-1} - 1]\big)$$
$$= A_m\big(A_{m-1}[1, f_{m-1} - 1]\big)$$
$$= \big(A_m A_{m-1}\big)[1, f_{m+1} - 1]$$
$$\subset \big(A_m A_{m-1} \cup A_{m-1}A_m\big)[1, f_{m+1} - 1]$$
$$= A_{m+1}[1, f_{m+1} - 1],$$

which completes the induction.

Now let us turn to the case with $k \geq 1$. We shall also prove this by induction. We proved the basis step, $k = 1$, above. Now we assume that (8) holds for $k = p$. For $k = p + 1$ we have by what we proved above

$$A_{n+p}[1, f_{n+p} - 1] = A_{n+p+1}[1, f_{n+p} - 1].$$

But then also $A_{n+p}[1, f_n - 1] = A_{n+p+1}[1, f_n - 1]$ and therefore by our induction assumption

$$A_n[1, f_n - 1] = A_{n+p}[1, f_n - 1] = A_{n+p+1}[1, f_n - 1],$$

which completes the proof of (8). Equality (9) is direct from (8) by symmetry. $\qquad\square$

Proposition 8. *For $n \geq 4$ we have*

$$A_n \subset \big(A_{n-1}[1, f_{n-1} - 1]\big)\{0, 1\}^2 \big(A_{n-2}[2, f_{n-2}]\big). \tag{10}$$

Proof. Denote the right hand side set in (10) by B_n. If $n = 4$ we have $B_4 = (A_3[1,1])\{0,1\}^2 = \{0,1\}^3$, which clearly is a superset of A_4. If $n \geq 5$ then by Proposition 7 we have $A_n[1, f_{n-1}] = B_n[1, f_{n-1}]$ and $A_n[f_{n-1} + 2, f_n] = A_{n-2}[2, f_{n-2}] = B_n[f_{n-1} + 2, f_n]$. As B_n by definition consist of all words starting with a prefixes in $A_n[1, f_{n-1}]$ and ending with a suffix in $A_n[f_{n-1} + 2, f_n]$ it must contain all sequences from A_n. $\square$

Note that there clearly is a symmetric analogous of the result of Proposition 8.

4 FACTOR SETS

The aim of this section is to give a finite method for finding the factor set F_n. For some small values of n we have

$$F(A_3, f_2) = \{0, 1\}, \qquad F(A_4, f_3) = \{01, 10, 11\},$$

$$F(A_5, f_4) = \{001, 010, 011, 100, 101, 110, 111\},$$

$$F(A_6, f_5) = \left\{ \begin{array}{l} 00101, 00110, 00111, 01001, 01010, \\ 01011, 01100, 01101, 01110, 01111, \\ 10010, 10011, 10100, 10101, 10110, \\ 10111, 11001, 11010, 11011, 11100, \\ 11101, 11110 \end{array} \right\}.$$

We shall shortly see that for $n \geq 4$ the above sets $F(A_{n+1}, f_n)$ actually is F_n. It is clear from above that $F_3 \neq F(A_4, f_3)$ as words in $F(A_5, f_4)$ have the sub-word 00, while this sub-word is not in $F(A_4, f_3)$. By computer calculation we can find the size of $F(A_{n+1}, f_n)$ for some additional small values of n, see Table 1 on page 11.

Proposition 9. *For $k \geq 1$ and $n \geq 4$ we have*

$$F(A_{n+1}, f_n) = F(A_{n+k}, f_n). \tag{11}$$

Proof. We give a proof by induction on k. For the basis step, $k = 2$, we see that $F(A_{4+1}, f_4) = F(A_{4+2}, f_4)$, (this is direct as the word 000 is not a sub-word of an element in $F(A_6, f_5)$). Hence we may assume that $n \geq 5$. From the recursion (2) we have directly that $F(A_{n+1}, f_n) \subset F(A_{n+2}, f_n)$. For the reversed inclusion we have by the recursion identity (2) and symmetry that it is enough to show that any sub-word of length f_n in $A_{n+1} A_n$ is also a sub-word in A_{n+1}. Let us define $x_i = x[i, i - 1 + f_n]$ where $x \in A_{n+1} A_n$ and $1 \leq i \leq f_{n+1} + 1$.

For $1 \leq i \leq f_{n-1} + 1$ we have directly that x_i is a sub-word in A_{n+1}.

For $f_{n-1} + 2 \leq i < 2f_{n-1}$ we have by the definition of x_i that it is a sub-word in $A_{n+1} A_n[f_{n-1} + 2, f_{n+1} + f_{n-1} - 1]$. (The inequalities defining

the case are valid if $f_{n-1} \geq 3$, which requires that $n \geq 5$). By Proposition 7 and the recursion (2) we have

$$
\begin{aligned}
\big(A_{n+1}A_n\big)[f_{n-1} + 2, f_{n+1} + f_{n-1} - 1] &= \\
&= \big(A_{n+1}[f_{n-1} + 2, f_{n+1}]\big)\big(A_n[1, f_{n-1} - 1]\big) \\
&= \big(A_n[2, f_n]\big)\big(A_{n-1}[1, f_{n-1} - 1]\big) \\
&= \big(A_n A_{n-1}\big)[2, f_{n+1} - 1] \\
&\subset \big(A_n A_{n-1} \cup A_{n-1}A_n\big)[2, f_{n+1} - 1] \\
&= A_{n+1}[2, f_{n+1} - 1],
\end{aligned}
$$

which gives that x_i is a sub-word of A_{n+1}.

For $2f_{n-1} \leq i \leq f_{n+1}$ we have that x_i is a sub-word in $\big(A_{n+1}A_n\big)[f_n + 2, f_{n+2}]$. Proposition 7 and the recursion (2) now gives

$$
\begin{aligned}
\big(A_{n+1}A_n\big)[f_n + 2, f_{n+2}] &= \big(A_{n+1}[f_n + 2, f_{n+1}]\big)A_n \\
&= \big(A_{n-1}[2, f_{n-1}]\big)A_n \\
&= \big(A_{n-1}A_n\big)[2, f_{n+1} - 1] \\
&\subset \big(A_n A_{n-1} \cup A_{n-1}A_n\big)[2, f_{n+1} - 1] \\
&= A_{n+1}[2, f_{n+1} - 1],
\end{aligned}
$$

and again we see that x_i is a sub-word of A_{n+1}.

For finally $i = f_{n+1} + 1$ we have that x_i is an element in A_n, and therefore also a sub-word in A_{n+1}, which concludes the proof of the basis step.

Assume for induction that equality (11) holds for $k = p$. For the induction step, $k = p + 1$, we have from what we just proved

$$
F(A_{n+p}, f_{n+p}) = F(A_{n+p+1}, f_{n+p}).
$$

But then we must also have $F(A_{n+p}, f_n) = F(A_{n+p+1}, f_n)$. The induction assumption now gives $F(A_n, f_n) = F(A_{n+p}, f_n) = F(A_{n+p+1}, f_n)$, which proves the proposition. $\qquad\square$

Proposition 10. *For $n \geq 4$ we have $F_n = F(A_{n+1}, f_n)$.*

Proof. It is clear that $F(A_{n+1}, f_n) \subset F_n$. For the converse let $x \in F_n$. Then there are two words u and v such that $uxv \in A_m$ for some $m \geq n + 1$, that is $x \in F(A_m, f_n)$. Proposition 9 now gives $F(A_m, f_n) = F(A_{n+1}, f_n)$. $\quad\square$

5 Upper Bound

The aim of this section is to give an upper bound of the size of $|F_n|$ in terms of $|A_n|$.

Proposition 11. *For $n \geq 3$ and $1 \leq k \leq f_n - 1$ we have*

$$
\big|A_n[1, k]\big| \cdot \big|A_n[k + 1, f_n]\big| \leq 4^{n-2}|A_n|. \tag{12}
$$

Proof. It is clear from the symmetric structure of A_n that we only have to give a proof for the case $1 \leq k \leq \lfloor \frac{1}{2} f_n \rfloor$. We give a proof by induction on n. For the basis case $n = 3$ we have

$$|A_3[1,1]| \cdot |A_3[2,2]| = |\{0,1\}| \cdot |\{0,1\}| = 2 \cdot 2 \leq 4 \cdot 2 = 4^{3-2}|A_3|.$$

Similarly we have for $n = 4$

$$|A_4[1,1]| \cdot |A_4[2,3]| \leq |A_4| \cdot |A_4| = 3 \cdot |A_4| \leq 4^{4-2}|A_4|$$

and for $n = 5$ with $k = 1, 2$

$$|A_5[k,1]| \cdot |A_5[k+1,5]| \leq |A_5| \cdot |A_5| = 8 \cdot |A_5| \leq 4^{5-2}|A_5|.$$

Now assume for induction that (12) holds for $5 \leq n \leq p$. For the induction step, $n = p + 1$, we have from Proposition 7, Proposition 8, the induction assumption and (5) that

$$
\begin{aligned}
\left|A_{p+1}[1,k]\right| \cdot \left|A_{p+1}[k+1, f_{p+1}]\right| &= \\
&= \left|A_p[1,k]\right| \cdot \left|A_{p+1}[k+1, f_{p+1}]\right| \\
&\leq \left|A_p[1,k]\right| \cdot \left|\left(\left(A_p[1, f_p - 1]\right)\{0,1\}^2\left(A_{p-1}[2, f_{p-1}]\right)\right)[k+1, f_{p+1}]\right| \\
&= \left|A_p[1,k]\right| \cdot \left|\left(A_p[k+1, f_p - 1]\right)\{0,1\}^2\left(A_{p-1}[2, f_{p-1}]\right)\right| \\
&\leq \left|A_p[1,k]\right| \cdot \left|A_p[k+1, f_p]\right| \cdot 2^2 \cdot \left|A_{p-1}[2, f_{p-1}]\right| \\
&\leq 4^{p-2} \cdot |A_p| \cdot 4 \cdot |A_{p-1}| \\
&\leq 4^{(p+1)-2}|A_{p+1}|,
\end{aligned}
$$

which completes the induction. $\square$

The estimate in Proposition 11 seems to be far off from the true upper bound of the left hand side in (12). If we let

$$c_n = \max_{1 \leq k \leq f_n - 1} \frac{\left|A_n[1,k]\right| \cdot \left|A_n[k+1, f_n]\right|}{|A_n|}$$

then computer calculations (see Table 1 on page 11) motivate the following conjecture

Conjecture 12. *There is a constant C such that for all $n \geq 3$ and $1 \leq k \leq f_n - 1$ we have*

$$\left|A_n[1,k]\right| \cdot \left|A_n[k+1, f_n]\right| \leq C \cdot |A_n|.$$

From our computer calculations (see Table 1 on page 11) we further conjecture that we may find $C < 3$, but we have to leave this question open as well as the more delicate question of finding the optimal value of such a constant, $c_o = \sup_{n \geq 3} c_n$.

We can now give the estimate we set out to find. Compared to computer calculations our estimate seems to be far from the optimal one, but it will be sufficient for our purpose.

Proposition 13. *For $n \geq 3$ we have*

$$|F_n| \leq 2 \left(4^{n-2} f_{n-1} + 1\right) |A_n|.$$

Proof. For $n = 3$ we have

$$|F_3| = 4 \leq 2(4 \cdot 1 + 1) \cdot 2 = 2(4^{3-2} f_{3-1} + 1)|A_3|.$$

For $n \geq 4$ we have from Proposition 10 that $F_n = F(A_{n+1}, f_n)$. Therefore it is enough to give an estimarte of the number of subwords of length f_n there are in the words in A_{n+1}. Let us start by considering the number of subwords in the subset $A_n A_{n-1}$. We define $x_k = x[k, k-1+f_n]$, where $x \in A_n A_{n-1}$ for $1 \leq k \leq f_{n-1} + 1$. Clearly we have $x_k \in F_n$. If $k = 1$ then we have $x_1 \in A_n$ and we see that there are $|A_n|$ different x_1's. If $2 \leq k \leq f_{n-1} + 1$ we have by Proposition 7

$$x_k \in (A_n[k, f_n])(A_{n-1}[1, k-1]) = (A_n[k, f_n])(A_n[1, k-1]).$$

Proposition 11 now gives that there are at most $4^{n-2}|A_n|$ different x_k's for each of these k's. Adding up the estimates and include the symmetric case, $x \in A_{n-1}A_n$, completes the proof. □

6 PROOF OF THE THEOREM

Finally, we have now gathered enough background to prove our theorem

Proof of Theorem 3. We have already dealt with the existence of both limits in (3). Since clearly $A_n \subset F_n$ it now follows from Proposition 13,

$$\begin{aligned}
\lim_{n \to \infty} \frac{\log |A_n|}{f_n} &\leq \lim_{n \to \infty} \frac{\log |F_n|}{f_n} \\
&\leq \lim_{n \to \infty} \frac{\log \left(2(4^{n-2} f_{n-1} + 1)|A_n|\right)}{f_n} \\
&\leq \lim_{n \to \infty} \frac{\log \left(4^{n-1} 2^n |A_n|\right)}{f_n} \\
&\leq \lim_{n \to \infty} \frac{3n \log 2}{f_n} + \lim_{n \to \infty} \frac{\log |A_n|}{f_n} \\
&= \lim_{n \to \infty} \frac{\log |A_n|}{f_n},
\end{aligned}$$

which completes the proof. □

7 Numerics

We present here the output from our computer calculations, which were obtained by a small JAVA-program on a standard PC.

Table 1

n	f_n	$|A_n|$	$|F_n|$	$|F(A_{n+1}, f_n)|$	c_n
0	0	0			
1	1	1	2	1	
2	1	1	2	2	
3	2	2	4	3	2.0
4	3	3	7	7	2.0
5	5	8	22	22	2.0
6	8	30	108	108	2.13333
7	13	288	1356	1356	2.11111
8	21	10080	65800	65800	2.17143
9	34	3317760	30139200	30139200	2.16389

8 Acknowledgment

References

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